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$\mathcal{N} = 1^*$ vacua, fuzzy spheres and integrable systems

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Abstract: We calculate the exact eigenvalues of the adjoint scalar fields in the massive vacua of $\mathcal{N} = 1^*$ SUSY Yang-Mills with gauge group SU(N). This provides a field theory prediction for the distribution of D3-brane charge in the AdS-dual. We verify the proposal of Polchinski and Strassler that the D3-brane's lie on a fuzzy sphere in the supergravity limit and determine the corrections to this distribution due to world-sheet and quantum effects. The calculation utilizes the connection between SUSY gauge theories and classical integrable systems. The analysis also provides several new results concerning the equilibrium configurations of the N-body Calogero-Moser hamiltonian.

Keywords: Brane Dynamics in Gauge Theories, AdS/CFT and dS/CFT Correspondence, Supersymmetric Effective Theories.

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Contents

1. Introduction

The $\mathcal{N} = 1^*$ deformation of $\mathcal{N} = 4$ SUSY Yang-Mills theory with gauge group SU(N) provides a fascinating example of a non-conformal gauge theory where a direct comparison of the AdS-dual with field theory results is possible. Specifically, as this is a theory with $\mathcal{N} = 1$ SUSY, there are special observables which depend holomorphically on the parameters and can be computed exactly for all values of the marginal coupling $\tau = 4\pi i/g^2 + \theta/2\pi$ and N. In the limit of large 't Hooft coupling, $\lambda = g^2N/4\pi \to \infty$, the results can be compared with a supergravity computation in the IIB-dual proposed by Polchinski and Strassler [18].

The $\mathcal{N} = 1^*$ theory contains three complex scalar fields Φ_1, Φ_2 and Φ_3, in the adjoint representation of $G = \mathrm{SU}(N)$. After a rescaling, these fields appear on an equal footing and may be rotated into each other by an SO(3) subgroup of the SU(4) R-symmetry of the $\mathcal{N} = 4$ theory. In the following we will study the eigenvalues of these $N \times N$ matrices in supersymmetric vacuum states of the theory, or equivalently, the vacuum expectation values of the $N - 1$ gauge-invariant operators $u_k = \mathrm{Tr}_N\, \Phi^k$ for $k = 2, \dots, N$. Here, and in the following, Φ denotes any one of the three Φ_i and λ_k, with $k = 0, 1, \dots, N-1$, denotes its eigenvalues. The main result of this paper is an exact formula for the large-N distribution of these eigenvalues in each massive vacuum state of the theory. As we explain below, the desired result may be obtained by considering the equilibrium configurations of a certain classical integrable system. In the remainder of this introductory section we will give this result and discuss its physical interpretation. Details of the calculations are provided in subsequent sections.

We begin by reviewing the classical vacuum structure of the theory. The classical F-term vacuum equations are solved when the three adjoint scalars (appropriately rescaled) obey an SU(2) algebra $[\Phi_i, \Phi_j] = \varepsilon_{ijk}\Phi_k$. In fact, as first explained in [1], the classical

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theory has one SUSY vacuum for each N-dimensional representation of SU(2), with the VEVs of the Φ_i being given by the anti-hermitean generators acting in this representation. An arbitrary representation of dimension N can be decomposed as the sum of irreducible representations whose dimensions add up to N. The total number of classical vacua is therefore equal to the number of partitions of N. The unique irreducible representation corresponds to a vacuum where the gauge group is completely broken by the Higgs mechanism and the spectrum is massive. We will refer to this as the "Higgs vacuum". In vacua corresponding to reducible representations, a non-trivial subgroup of SU(N) is left unbroken and there are massless particles at the classical level.

This correspondence between classical $\mathcal{N}=1^*$ vacua and SU(2) representations has an interesting realization in weakly coupled IIB string theory where the undeformed $\mathcal{N}=4$ theory with gauge group U(N) lives on the worldvolume of N parallel D3-branes. The $\mathcal{N}=1^*$ deformation corresponds to turning on the flux of the two-form antisymmetric tensor field of the Ramond-Ramond sector of the IIB theory. The background RR flux causes the D3-branes to polarize to spherically wrapped D5-branes by the Myers effect [6]. As there is non-zero D3-brane charge on the two wrapped dimensions of the fivebrane, the corresponding worldvolume theory is non-commutative. In field theory, the non-commutivity is realized by the SU(2) algebra described above satisfied by the VEVs of the adjoint scalar fields, Φ_i. The N-dimensional irreducible representation corresponding to the Higgs vacuum is the familiar matrix representation of the fuzzy sphere. Reducible representations correspond to multiple concentric fuzzy spheres in an obvious way.

With the standard normalization of the generators, the eigenvalues of each of the adjoint scalars in the Higgs vacuum is uniformly distributed on the imaginary axis between $\pm i(N-1)/2$ with unit spacing. Explicitly, the N eigenvalues are $\lambda_k = i(N-2k-1)/2$ with $k=0,1,N-1$. In section 2.2, we will derive an exact quantum mechanical formula for the eigenvalues λ_k in the Higgs vacuum (see eq. (2.25)). In the large-N limit we replace k/N by a continuous variable $0 \leq x < 1$. The range of the eigenvalues determines the radius of the sphere on which the D3-branes lie. Equivalently, it is also determined by the expectation value of the lowest non-zero condensate u_2 which has already been studied in detail in earlier work [11, 13, 14]. For the present purposes it is convenient to normalize the range of the eigenvalues to unity, defining a normalized eigenvalue distribution $\tilde{\lambda}(x)$. (The natural normalization as we later show involves the dual 't Hooft coupling and the modular weight.) The large-N limit of the semiclassical result therefore yields a uniform distribution $\tilde{\lambda}(x) = 1 - x/2$. In the following we will see that this uniform distribution also emerges in the strong-coupling supergravity regime.

So far we have only described the classical limit of the $\mathcal{N}=1^*$ theory. In vacua with an unbroken non-abelian gauge symmetry, the low-energy theory is realized in a strongly-coupled confining phase with gluino condensation and a mass gap. The vacuum structure of the quantum theory is determined by the exact superpotential derived in [11].

$$\mathcal{W} = m_1 m_2 m_3 \sum_{a>b} \mathcal{P}\left(X_a - X_b\right). \tag{1.1}$$

This result has recently been extended to other gauge groups in [12]. Here, $\mathcal{P}(X)$ is the

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Weierstrass function and m_i, $i = 1, 2, 3$, are the masses of the three chiral multiplets which we will set to one from now on. The N auxiliary variables X_a, with $a = 1, 2, \ldots, N$, only have a direct physical interpretation when the theory is compactified on a circle down to three dimensions. In this context they can be identified with a complex combination of the Wilson and 't Hooft loops of the gauge field around the compact direction. These variables naturally take values on a torus whose complex structure is the complexified coupling constant τ, and the exact superpotential, being an elliptic function, is single-valued on this torus. The S-duality transformations of the $\mathcal{N} = 4$ theory correspond to modular transformations of the torus.

Stationarizing the superpotential with respect to the N variables X_a, we find a set of massive vacua labeled by integers p,q and k, with $pq = N$ and $k = 0, \ldots, q-1$. The total number of these vacua is the sum of the divisors of N. For each divisor q of N, the q quantum vacua labelled by $k = 0, \ldots, q-1$ reduce in the weak coupling limit to the unique classical vacuum where the unbroken gauge symmetry is SU(q). This classical vacuum corresponds to the reducible representation of SU(2) which is the direct sum of q copies of the irreducible representation of dimension $p = N/q$. The Higgs vacuum corresponds to $q = 1$, $p = N$. For $q > 1$, the low energy theory is strongly coupled and gluino condensation occurs, splitting the massless classical vacuum into q massive quantum vacua. The theory also has a large number of vacua which remain massless at the quantum level but we will not consider these here.

A remarkable feature of the $\mathcal{N} = 1^*$ theory is that the set of $\sum_{d|N} d$ vacua described above are transformed into each other by the S-duality of the underlying $\mathcal{N} = 4$ theory. In particular, the electromagnetic duality transformation $\tau \to -1/\tau$ interchanges the Higgs vacuum in which electric charges condense and a confining vacuum in which magnetic charges condense. This provides a concrete realization of the duality between Higgs and confining phases conjectured many years ago by 't Hooft. This property is manifest in the exact results following from the superpotential (1.1). The action of S-duality can also be understood in the IIB realization of the theory described above. As above the Higgs vacuum is realized on N parallel D3-branes polarized into a spherically wrapped D5-brane. The D3-branes are invariant under the SL$(2, \mathbb{Z})$ duality of IIB string theory, but the D5-brane is not. The action of S-duality generates an SL$(2, \mathbb{Z})$ orbit of vacuum configurations described by (m, n)-fivebranes wrapped on non-commutative two-spheres each carrying N units of D3-brane charge. Here m and n are determined in terms of the the integers p, q and k which describe the field theory vacua [18].

Polchinski and Strassler studied the AdS-duals of these $\mathcal{N} = 1^*$ vacua. They proposed a dual-IIB geometry for each vacuum which is asymptotically $AdS_5 \times S^5$ near the boundary, but is deformed by the presence of the spherically wrapped (m, n)-fivebranes in the interior. The precise domain in which the supergravity approximation is reliable depends explicitly on the choice of vacuum. We start from the vacuum which exhibits pure magnetic confinement, corresponding to the choice $p = 1$, $q = N$ and $k = 0$. This vacuum is described by a single spherically wrapped NS five-brane. The SUGRA description is valid when the sphere is large in string units and this is true as long as the "electric" 't Hooft coupling, $\lambda = g^2 N/4\pi$, is much larger than one. Away from this limit corrections from IIB worldsheet

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instantons wrapping the S^2 are unsuppressed. Part of the charm of this model is that these corrections can be calculated explicitly in the dual field theory. As explained above the remaining vacua are related to the magnetic confinement vacuum by S-duality. Specifically, we perform an $\mathrm{SL}(2,\mathbb{Z})$ transformation, $\tau \to \tau_D = (a\tau+b)/(c\tau+d) = 4\pi i/g_D^2 + \theta_D/2\pi$ where a, b, c, d are integers with $ad-bc=1$ which transforms the wrapped NS five-brane into an (a,c) five-brane. In this vacuum we define a dual 't Hooft coupling $\lambda_D = g_D^2 N/4\pi$ and the classical supergravity approximation is reliable whenever $\lambda_D \gg 1$ with $g_D^2 \ll 1$. Away from large λ_D, corrections coming from (a,c)-strings wrapping the two-sphere become significant.

In the regime described above, the supergravity dual can be used to compute various field theory quantities such as the tensions of BPS domain walls and the VEVs of chiral operators. In [13, 18], these were successfully compared with the exact field theory results of [11, 14]. However, the particular quantities tested so far correspond to operators of low R-charge such as the gluino condensate and the operator u_2 defined above. These operators correspond to massless scalars present in five-dimensional gauged supergravity. One of the main aims of this paper is to extend this test to operators of higher R-charge corresponding to Kaluza-Klein modes which probe the full ten-dimensional geometry. Specifically, the eigenvalues λ_k determine the VEVs of the operators u_k, which have charge proportional to k under a U(1) subgroup of the SU(4) R-symmetry of the $\mathcal{N}=4$ theory. The corresponding SUGRA states live in the tower of spherical harmonics on the five-sphere of the UV geometry.

The VEVs of the operators u_k can be read off from the asymptotic behavior of these SUGRA fields in the Polchinski-Strassler solution corresponding to each vacuum. Indeed, this was the procedure followed in [13] to compute the VEV of u_2. However it is simpler to note that, as at weak-coupling, the eigenvalues λ_k determine the position of the distribution of D3-brane charge in spacetime. The correspondence between these two points of view for the $\mathcal{N}=4$ theory is discussed in [5]. By calculating the λ_k directly in field theory and then taking the SUGRA limit, we will be able to check directly that the 3-branes lie on non-commutative two-spheres in spacetime as predicted by Polchinski and Strassler. Our main result is a universal formula for the large-N eigenvalue distribution which is valid in all the massive vacua

$$\tilde{\lambda}(x) = \frac{1}{2} - x + \sum_{l=1}^{\infty} \left(\frac{e^{-2\pi\lambda_D(l-x)}}{1+e^{-2\pi\lambda_D(l-x)}} - \frac{e^{-2\pi\lambda_D(l-1+x)}}{1+e^{-2\pi\lambda_D(l-1+x)}} \right). \tag{1.2}$$

Here, as above $\lambda_D = g_D^2 N/4\pi$ is the appropriate 't Hooft coupling in each vacuum. In the limit $\lambda_D \to \infty$ only the first term on the right-hand side contributes and we find a uniform distribution of eigenvalues centered at the origin $x=0$. This is consistent with the fuzzy sphere structure described above. In particular, we recall that the eigenvalues of the generators of SU(2) in an irreducible representation are uniformly distributed around the origin. The corrections correspond to the contributions of wrapped (a,c)-strings around this sphere [13, 18].

Our approach to determining the exact eigenvalues λ_k is based on the connection between supersymmetric gauge theories and integrable systems first discovered in [2, 3, 4]. In particular, it is noteworthy that the exact superpotential coincides with the potential

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energy of the elliptic Calogero-Moser system. The latter is an integrable model describing non-relativistic particles moving in the two-body interaction potential determined by the Weierstrass potential. In terms of the positions and momenta of the N particles, X_a and P_a with $a = 1, \ldots, N$, the hamiltonian is,

$$H = \sum_{a=1}^{N} \frac{P_a^2}{2} + \sum_{a>b} \mathcal{P}\left(X_a - X_b\right). \tag{1.3}$$

Although the positions and momenta are real in the first instance, it is convenient to promote them to complex variables.

In its original form the connection between integrable systems and supersymmetric gauge theory involves theories with $\mathcal{N} = 2$ supersymmetry. The theory relevant to the present case is the $\mathcal{N} = 2$ supersymmetric deformation of the $\mathcal{N} = 4$ theory, also known as the $\mathcal{N} = 2^*$ theory. In particular the complex curve which governs the Coulomb branch of this theory coincides with the *spectral curve* of the Calogero-Moser system which plays a key role in its explicit integration. As explained in [11], the fact that the exact superpotential of the $\mathcal{N} = 1^*$ theory coincides with the Calogero-Moser hamiltonian follows from soft breaking to $\mathcal{N} = 1$ supersymmetry. In particular, the supersymmetric vacua are in one to one correspondence with the equilibrium positions of the (complexified) Calogero-Moser system. The results given below extend the existing results in the literature about the equilibrium properties of the integrable system. In particular, they provide an elliptic extension of the results given in [8] for the trigonometric Calogero-Sutherland model.

For the present purposes, certain aspects of the correspondence between the $\mathcal{N} = 1^*$ system and the Calogero-Moser system are particularly useful. The integrability of the Calogero-Moser system is equivalent to the existence of N conserved quantities or hamiltonians, H_k with $k = 1, 2, \ldots N$, of which the first two are the total momentum and energy. As discussed in the next section, hamilton's equations of motion for the system (1.3) have a Lax Representation and the N conserved quantities are determined by the eigenvalues of the *Lax matrix* L as $H_k = \mathrm{Tr}_N L^k$. On the other hand, the identification of the Calogero-Moser spectral curve with the complex curve of the $\mathcal{N} = 2^*$ theory reveals that the hamiltonians H_k should be identified with the VEVs of the operators[1] $u_k = \mathrm{Tr}_N \Phi^k$. This gives us a very convenient way to calculate the eigenvalues λ_k in each SUSY vacuum: they can be identified with the eigenvalues of the Lax matrix in the corresponding equilibrium position of the Calogero-Moser system.

The identification described above is complicated by the problem of operator mixing as discussed in [13]. In particular, once masses are turned on, the operator u_j can mix with the other operators u_k with $k < j$ which have lower dimensions. This is particularly clear in the $\mathcal{N} = 2^*$ theory, where the ambiguity is simply that of choosing holomorphic coordinates on the Coulomb branch. Similarly, in the integrable system, it is possible to find many inequivalent Lax matrices whose eigenvalues correspond to different linear combinations of the conserved quantities. Hence there is no unique way to identify the

[1] The vanishing of u_1 in the $\mathrm{SU}(N)$ theory corresponds to choosing the center of momentum frame $H_1 = \sum_{a=1}^{N} p_a = 0$ in the integrable model.

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operators u_k in terms of the conserved hamiltonians. Fortunately this ambiguity turns out to be very mild in the large-N limit. As explained in [13] the ambiguity is vacuum independent and can be fixed by determining the mixing coefficients in any given vacuum. These are holomorphic functions of the complexified coupling which necessarily have a weak coupling expansion in integer powers of the Yang-Mills instanton factor $q_{\rm YM} = \exp 2\pi i\tau$. These terms vanish exponentially in the 't Hooft limit, and the mixing coefficients are pure numbers which can be determined by a semiclassical calculation. The upshot is that once we have determined the semiclassical relation between the eigenvalues of the Lax matrix and the field theory operators in the Higgs vacuum we can make definite statements about the large-N distribution of D3-branes in the other vacua.

The paper is organized as follows. In section 2 we introduce the integrable system and determine the eigenvalues of the Lax matrix in the equilibrium configurations. In section 3, we check the prediction of Polchinski and Strassler that the eigenvalues lie on a fuzzy sphere in each vacuum. Section 4 we perform a detailed analysis of the results in the confining vacuum and in section 5 we compare with earlier results for the condensate u_2. Another approach to calculating the field theory VEVs, based on the complex curve of the $\mathcal{N} = 2^*$ theory is described in an appendix.

2. The Calogero-Moser integrable system

The CM system [7] consists of N non-relativistic particles with position X_a which interact via the two-body potential

$$V = \sum_{a>b} \mathcal{P}(X_a - X_b)\,, \qquad a, b = 1 \ldots N\,, \tag{2.1}$$

where $\mathcal{P}$ is the Weierstrass function defined on a torus with periods $2\omega_1 = 2i\pi$ and $2\omega_2 = 2i\pi\tau$. The coordinates and momenta are considered complexified and the positions of particles are centered so that

$$\sum_{a=1}^{N} X_a = 0\,. \tag{2.2}$$

The system has a non-trivial equilibrium position at,

$$X_a = \frac{2i\pi a}{N} - \frac{i\pi(N-1)}{N}\,, \qquad a = 1 \ldots N\,, \tag{2.3}$$

where the N particles are equally spaced along the one-cycle of the torus parallel to the period $2\omega_1$. As explained in [11], this configuration corresponds to the Higgs vacuum of the $\mathcal{N} = 1^*$ theory. Acting by modular transformations, we find a family of equilibrium configurations equal in number to the sum of the divisors of N less one where the particles are equally space along different cycles of the torus. These correspond to field theory vacua in the confining phase.

The Calogero-Moser system can be described in terms of a pair of Lax matrices, and is characterized by N integrals of motion and corresponding angular variables. The $\mathcal{N} = 1^*$ vacua are described in terms of the N order parameters in the moduli space $u_k = {\rm Tr}\Phi^k$,

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which correspond to the N integrals of motion of the integrable system modulo the potential ambiguities described above. The integrals of motions or hamiltonians for the system are the symmetric polynomials of the eigenvalues of the Lax matrix L.

Hamilton's equations for a system of N particles are equivalent to the Lax matrix equation [7]

$$i\dot{L} = [M, L]\,, \tag{2.4}$$

where

$$\begin{aligned} L_{ab} &= p_a\delta_{ab} + i(1-\delta_{ab})x(X_a - X_b)\,, \\ M_{ab} &= \delta_{ab}\sum_{a\neq c} z(X_a - X_c) - (1-\delta_{ab})y(X_a - X_b) \end{aligned} \tag{2.5}$$

with

$$\begin{aligned} V(u) &= x(u)^2 + C\,, \\ y(u) &= -x'(u)\,, \\ z(u) &= \frac{x''(u)}{2x(u)}\,, \end{aligned} \tag{2.6}$$

where C is a constant. In the present case where the potential V is the Weierstrass function, $x(u)$ is given in terms of jacobi elliptic functions [7]. There is more then one possible choice for the function $x(u)$. Different choices lead to different possible Lax matrices whose eigenvalues lead to distinct linear combinations of the N conserved quantities. A particularly convenient choice is,

$$x(u) = \frac{\mathbf{K}}{i\pi} cs\left(\frac{\mathbf{K}u}{i\pi}\right)\,, \tag{2.7}$$

where $cs(u)$ is defined as the ratio of the jacobian elliptic functions $cs(u) = \mathcal{N}(u)/sn(u)$, and $\mathbf{K}$ and $\mathbf{K}'$ are complete elliptic integrals. The square of $x(u)$ is periodic in u with periods

$$\begin{aligned} 2\omega_1 &= 2\pi i\,, \\ 2\omega_2 &= -\frac{2\pi\mathbf{K}'}{\mathbf{K}} = 2\pi i\tau \end{aligned} \tag{2.8}$$

so

$$\frac{\mathbf{K}'}{\mathbf{K}} = -i\tau\,. \tag{2.9}$$

The Lax matrix in the Higgs vacuum equilibrium position has the form

$$L_{ab} = (1-\delta_{ab})\frac{\mathbf{K}}{\pi} cs\left(\frac{\mathbf{K}u_{ab}}{i\pi}\right)\,, \tag{2.10}$$

where

$$u_{ab} = \frac{2\pi i}{N}(a-b)\,. \tag{2.11}$$

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Conveniently, the matrix L is a circulant, and its eigenvalues are computed by a standard formula as,

$$
\begin{aligned}
\lambda_a &= \sum_{m=1}^{N-1} A_m z_a^m \,, \\
A_m &= -\frac{\mathbf{K}}{\pi} cs\left(\frac{2\mathbf{K}m}{N}\right) \,, \\
z_a &= e^{2\pi i a/N} \,.
\end{aligned} \tag{2.12}
$$

Using the periodic properties of $cs(u)$, the eigenvalues can be rewritten in the form of a discrete Fourier transform

$$
\lambda_a = i \sum_{m=1}^{N-1} A_m \sin\frac{2\pi a m}{N} \,. \tag{2.13}
$$

In the large-N limit, this becomes a continuous Fourier transform, and the eigenvalues are the Fourier coefficients of the function $cs(u)$. Here we note that the Lax matrix is manifestly traceless.

2.1 Semiclassical limit

Before giving a general formula for the eigenvalues in the Higgs vacuum, we examine the semiclassical limit of the integrable system, $g \to 0$ limit $\tau \to i\infty$. In this limit potential becomes trigonometric,

$$
V = \sum_{a>b} \left(\frac{1}{4\sinh^2 \frac{1}{2}(X_a - X_b)} + \frac{1}{12} \right) . \tag{2.14}
$$

The only equilibrium configuration for the system coincides with the configuration corresponding to the Higgs vacuum in the general case. The remaining vacua decouple in this limit. The Lax matrix for this system [7] in the equilibrium position reduces to,

$$
L_{ab} = \frac{1}{2}(1 - \delta_{ab}) \cot\frac{\pi(a-b)}{N} \tag{2.15}
$$

with the Lax matrix function $x(u)$

$$
x(u) = \frac{1}{2}\coth\frac{u}{2} \,. \tag{2.16}
$$

We have again chosen the Lax matrix function $x(u)$ for which L is a circulant, and so is convenient for the computation for the eigenvalues. Comparing the form of the Lax matrix to the general case (2.10), the general CM system is recognized as the elliptic deformation of the integrable system with $1/\sinh^2$ potential (Calogero-Sutherland system).

The eigenvalues are computed as

$$
\lambda_a = \frac{1}{2} \sum_{m=1}^{N-1} A_m z_a^m \,, \tag{2.17}
$$

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where

$$
\begin{aligned}
A_m &= -\cot\frac{\pi m}{N}\,,\\
z_a &= e^{2\pi i a/N}
\end{aligned}
\tag{2.18}
$$

λ_a can also be rewritten as

$$
\begin{aligned}
\lambda_a &= -\frac{i}{2}\sum_{m=1}^{N-1}\cot\frac{\pi m}{N}\sin\frac{2\pi a m}{N} = -\frac{i}{2}(N-2a)\,, \qquad a = 1\ldots N-1\,,\\
\lambda_N &= 0\,.
\end{aligned}
\tag{2.19}
$$

The summation identity used here is the first one in a series of Diophantine identitities first given in [8]. In general these relations are highly non-trivial and considerations of special properties of the Lax matrix of the Calogero-Sutherland system actually provided the proof. In this particular case the result is not hard to prove by elementary means.[2] In the following we will derive elliptic deformations of these identities.

The above formula for the eigenvalues of the Lax matrix, nearly matches the classical field theory formula $\lambda_k = -i(N-2k-1)/2$. In semiclassical field theory, the N eigenvalues of Φ are evenly distributed along the imaginary axis with unit spacing centered around the origin. In (2.19), only $N-1$ eigenvalues are distributed in this way and there is an extra zero eigenvalue. This discrepancy is suppressed by a power of $1/N$ and the two eigenvalue distributions agree in the large-N limit. This supports our identification of the eigenvalues of the Lax matrix with the field theory VEVs. As explained in the introduction, this identification can now be used to yield predictions for the D3-brane distributions in the other vacua which are unambiguous in the large-N limit.

2.2 General formula for the eigenvalues

A general formula for the eigenvalues arises using the Fourier expansion of the function $cs(u)$,

$$
cs(u) = \frac{\pi}{2\mathbf{K}}\left[\cot\frac{\pi u}{2\mathbf{K}} - 4\sum_{n=1}^{\infty}\frac{q^{2n}}{1+q^{2n}}\sin\frac{n\pi u}{\mathbf{K}}\right]
\tag{2.20}
$$

with

$$
q = e^{-\pi\frac{\mathrm{K}'}{\mathrm{K}}} = e^{i\pi\tau}\,.
\tag{2.21}
$$

Note that $q_{\mathrm{YM}} = q^2$ is the factor associated with a single Yang-Mills instanton.

Substituting the expansion

$$
\lambda_a = -\frac{i}{2}\sum_{m=1}^{N-1}\cot\frac{\pi m}{N}\sin\frac{2\pi a m}{N} + 2\sum_{m=1}^{N-1}\sum_{n=1}^{\infty}\frac{q^{2n}}{1+q^{2n}}\sin\frac{2\pi n m}{N}e^{2\pi i a m/N}\,.
\tag{2.22}
$$

The first piece in the summation gives back the semiclassical result for the eigenvalues (2.19). The remaining terms are instanton corrections.

[2] An easy way to prove the relation is considering the difference of two consecutive eigenvalues.

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Next we evaluate the eigenvalues for the general case keeping the full series in the summation. Since the sum is absolutely convergent, the order of the summation is interchangeable. The double summation then can be performed by rewriting

$$\lambda_a = -\frac{i}{2}(N-2a) - i\sum_{n=1}^{\infty}\sum_{m=1}^{N-1}\frac{q^{2n}}{1+q^{2n}}\left(e^{\frac{2\pi i m}{N}(a+n)} - e^{\frac{2\pi i m}{N}(a-n)}\right). \tag{2.23}$$

Summing the resulting exponentials, the first summation selects

$$\begin{aligned} n &= lN - a\,, \qquad l \geq 1\,, \\ n &= a - lN\,, \qquad l \leq 0 \end{aligned} \tag{2.24}$$

and the sum becomes

$$\begin{aligned} \lambda_a &= -\frac{i}{2}(N-2a) - iN\left(\sum_{l=1}^{\infty}\frac{q^{2(lN-a)}}{1+q^{2(lN-a)}} - \sum_{l=-\infty}^{l=0}\frac{q^{2(a-lN)}}{1+q^{2(a-lN)}}\right) \\ &= -\frac{i}{2}(N-2a) - iN\sum_{l=1}^{\infty}\left(\frac{q^{(2l-1)N}q^{N-2a}}{1+q^{(2l-1)N}q^{N-2a}} - \frac{q^{(2l-1)N}q^{2a-N}}{1+q^{(2l-1)N}q^{2a-N}}\right). \end{aligned} \tag{2.25}$$

The summation is recognized as the expansion of the derivative of logarithmic jacobi theta function

$$\frac{\theta_3'(z|q)}{\theta_3(z|q)} = 2i\sum_{l=1}^{\infty}\left(\frac{q^{2l-1}e^{2iz}}{1+q^{2l-1}e^{2iz}} - \frac{q^{2l-1}e^{-2iz}}{1+q^{2l-1}e^{-2iz}}\right). \tag{2.26}$$

The final result of the summation can be written as

$$\lambda_a = -\frac{i}{2}(N-2a) - \frac{1}{2}N\frac{\theta_3'(z_a|\tilde{q})}{\theta_3(z_a|\tilde{q})} \tag{2.27}$$

with

$$\begin{aligned} \tilde{q} &= q^N = e^{Ni\pi\tau}\,, \\ z_a &= \frac{1}{2}(N-2a)\pi\tau\,. \end{aligned} \tag{2.28}$$

This is a general formula for the eigenvalues as a function of N and the effective coupling in the Higgs vacuum $\tilde{\tau} = N\tau$.

3. Confining vacuum

In this section we compute the eigenvalues in the unique vacuum of the theory which exhibits purely magnetic confinement. As reviewed in section 1, the confining vacuum in question is the image of the Higgs vacuum considered in the previous section under the electromagnetic duality transformation $\tau \to -1/\tau$. As the moduli u_k each have holomorphic modular weight k, the individual eigenvalues should transform with modular weight one.

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Under the S transformation the jacobi theta function transforms according to jacobi's formula

$$\theta_3(z|\tau) = (-i\tau)^{-1/2} e^{z^2/\pi i\tau} \theta_3\left(\frac{z}{\tau}\Big| -\frac{1}{\tau}\right). \tag{3.1}$$

Performing the S transformation including the modular weight we get for the eigenvalues in the confining vacuum

$$\lambda_a = \frac{i}{2\tau}(N - 2a) + \frac{N}{2\tau}\frac{\theta_3'(\tilde{z}_a|\tilde{q})}{\theta_3(\tilde{z}_a|\tilde{q})}, \tag{3.2}$$

where $\tilde{\tau} = \tau/N$ is the effective coupling in the confining vacuum, in terms of which $\tilde{q}$ and $\tilde{z_a}$ are defined as

$$\begin{aligned} \tilde{q} &= e^{-i\pi/\tilde{\tau}}, \\ \tilde{z}_a &= -\left(\frac{1}{2} - \frac{a}{N}\right)\frac{\pi}{\tilde{\tau}}. \end{aligned} \tag{3.3}$$

Notice that this result depends on τ only via the effective coupling constant $\tilde{\tau} = \tau/N$. In the large-N 't Hooft limit, $\tilde{q} = e^{-\pi\lambda}$. The eigenvalues of the scalar fields in the confining vacuum, thus depend on the 't Hooft coupling in a non-trivial way. For large-λ we are in a regime where the UV theory is well described by IIB SUGRA on $AdS_5 \times S^5$ and we can compare directly with the Polchinski-Strassler solution. However, the limit of small λ is also interesting. In this limit, we effectively decouple the adjoint hypermultiplet and flow to the minimal $\mathcal{N} = 2$ supersymmetric theory deformed by a mass term for the adjoint scalar.

In order to analyze the small λ limit, we need to use the modular properties of our result in the effective coupling $\tilde{\tau}$. In particular, performing the transformation $\tilde{\tau} \to -1/\tilde{\tau}$ and we find

$$\frac{\theta_3'(\tilde{z}_a| -\frac{1}{\tilde{\tau}})}{\theta_3(\tilde{z}_a| -\frac{1}{\tilde{\tau}})} = -\tilde{\tau}\frac{\theta_3'(-\tilde{\tau}\tilde{z}_a|\tilde{\tau})}{\theta_3(-\tilde{\tau}\tilde{z}_a|\tilde{\tau})} + \frac{2i\tilde{\tau}\tilde{z}_a}{\pi}. \tag{3.4}$$

Then the eigenvalues can be rewritten as

$$\lambda_a = -\frac{\theta_3'(-\tilde{\tau}\tilde{z}_a|\tilde{\tau})}{\theta_3(-\tilde{\tau}\tilde{z}_a|\tilde{\tau})}. \tag{3.5}$$

Here the anomalous piece of the θ functions under the $\tilde{S}$ transformation exactly cancels the leading constant term in λ_a. In the small λ limit $\tilde{\tau} \to i\infty$, $e^{i\pi\tilde{\tau}} \to 0$, and in this limit only the first term in the expansion of θ functions survives. Restoring the masses $m_1 = m_2 = M$, the combination $\Lambda = M\exp(-\pi/\lambda)$ can be idedntified with the dynamical scale of $\mathcal{N} = 2$ SUSY Yang-Mills theory with gauge group SU(N). The limiting formula for the eigenvalues is

$$\begin{aligned} \lambda_a &= 2\sin(2\tilde{\tau}\tilde{z}_a)e^{i\pi\tilde{\tau}} \\ &= -2e^{-\pi/\lambda}\sin\frac{2a\pi}{N}. \end{aligned} \tag{3.6}$$

In this limit, the eigenvalues computed from the Lax matrix can be compared with the results of Douglas and Shenker [16] for the minimal $\mathcal{N} = 2$ theory. In the large-N limit

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we can find the density ρ of the eigenvalues normalized to lie between zero and one. The result is the Wigner distribution

$$\rho(\lambda) = \left.\frac{1}{\lambda'(x)}\right|_{x=x(\lambda)} = \frac{2}{\pi\sqrt{1-\lambda^2}}\,. \tag{3.7}$$

This agrees with the large-N limit of the distribution found by Douglas and Shenker as computed in [17].[3]

4. General vacua in the supergravity limit

To compute the eigenvalue distributions in the supergravity limit, we first examine the limit in the confining vacuum. The eigenvalues in the confining vacua are given in (3.2), and in terms of N and the 't Hooft coupling λ can be explicitly written as

$$\lambda_a = \lambda\left[\frac{1}{2} - \frac{a}{N} + \sum_{l=1}^{\infty}\left(\frac{e^{-2\pi\lambda(l-\frac{a}{N})}}{1+e^{-2\pi\lambda(l-\frac{a}{N})}} - \frac{e^{-2\pi\lambda(l-1+\frac{a}{N})}}{1+e^{-2\pi\lambda(l-1+\frac{a}{N})}}\right)\right]. \tag{4.1}$$

The supergravity limit is taken $N \to \infty$, while keeping the 't Hooft coupling λ fixed, large. Taking the $N \to \infty$ limit k/N becomes a continuous parameter ranging from 0 to 1. In this limit we have the continuous distribution

$$\lambda(x) = \lambda\left[\frac{1}{2} - x + \sum_{l=1}^{\infty}\left(\frac{e^{-2\pi\lambda(l-x)}}{1+e^{-2\pi\lambda(l-x)}} - \frac{e^{-2\pi\lambda(l-1+x)}}{1+e^{-2\pi\lambda(l-1+x)}}\right)\right]. \tag{4.2}$$

When λ is large the summation terms are exponentially suppressed, so in the supergravity limit the the confining vacua are

$$\lambda(x) = \lambda\left(\frac{1}{2} - x\right) \tag{4.3}$$

corresponding to a constant distribution. (We note that at the endpoints $x = 0$ and $x = 1$ there is a constant shift in the eigenvalues. A similar endpoint effect can be seen when analyzing the eigenvalue distribution from the Seiberg-Witten curve. Such a constant endpoint shift does not effect the constant distribution of the eigenvalues.)

It is convenient to normalize the uniform distribution to range one, so in the following we will consider the suitably normalized eigenvalues. In the confining vacuum this means dividing by the 't Hooft coupling λ, so the distribution we arrive at the supergravity limit is

$$\tilde{\lambda}(x) = \frac{1}{2} - x\,, \tag{4.4}$$

where the normalized eigenvalues are denoted by $\tilde{\lambda}$.

Next we examine the supergravity limit of the eigenvalues in a general vacuum. The set of vacua are labelled with three integers (p, q, l) with the coupling constant

$$\tilde{\tau} = \frac{p\tau + l}{q}\,, \qquad pq = N\,, \qquad l = 0\ldots q-1\,. \tag{4.5}$$

[3] Our normalization is such that $\int_0^1 \rho(x)dx = 1$.

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The Higgs vacuum corresponds to $(N, 1, 0)$ while the confining vacuum is $(1, N, l)$. An $\mathrm{SL}(2, \mathbb{Z})$ transformation performed on a generic vacuum takes it into another vacuum, and also changes the coupling constant to a different coupling $\bar{\tau}$. Then there exists another $\mathrm{SL}(2, \mathbb{Z})$ transformation which maps this vacuum to one element of the sets of the original vacua, characterized with different integers (p', q', l'). In this way $\mathrm{SL}(2, \mathbb{Z})$ transformations map the set of vacua among themselves.

A general vacuum is th then connected to the confining vacuum by a modular transformation with weight 1:

$$\lambda(\tau) = \frac{1}{c\tau + d} \lambda_{\mathrm{conf}} \left(\frac{a\tau + b}{c\tau + d} \right) . \tag{4.6}$$

The weight can be seen for example from the description of vacua from the curve, in which the Riemann zeta functions transform with modular weight 1.

In a general vacuum the supergravity limit is taken by taking N to infinity, while keeping the dual 't Hooft coupling fixed and large. For a general vacuum connected to the Higgs vacuum by a modular transformation (4.6) we find in the large-N fixed λ_D limit

$$\tilde{\lambda}(x) = \frac{1}{2} - x + \sum_{l=1}^{\infty} \left(\frac{e^{-2\pi\lambda_D(l-x)}}{1 + e^{-2\pi\lambda_D(l-x)}} - \frac{e^{-2\pi\lambda_D(l-1+x)}}{1 + e^{-2\pi\lambda_D(l-1+x)}} \right) , \tag{4.7}$$

where λ_D is defined from $\tau_D = \frac{a\tau+b}{c\tau+d}$ as $\lambda_D = iN/\tau_D$. Here we normalized the eigenvalues appropriately by $\lambda_D/(c\tau+d)$. The modular factor comes as the weight of the weight-1 modular transformation. In the supergravity limit the summation is exponentially suppressed, and the eigenvalues for the general vacua are

$$\tilde{\lambda}(x) = \frac{1}{2} - x \, . \tag{4.8}$$

In conclusion, in the supergravity limit we find a uniform distribution for a general vacuum. The corrections to the supergravity limit are the summation terms with the exponential suppression in (4.7). These terms represent instanton contributions to the uniform supergravity distribution coming from (m, n)-strings which wrap the two-sphere in the Polchinski-Strassler geometry. The contribution of instantons can be rewritten by expanding the denominators and collecting the powers of each exponential term $e^{-\frac{2\pi\lambda}{N}m}$. In this way we arrive at the alternative expression

$$\tilde{\lambda}(x) = \frac{1}{2} - x + \sum_{m=1}^{\infty} \left(\sum_{\substack{d|m \\ d=-k \bmod N}} (-1)^{\frac{m}{d}-1} - \sum_{\substack{d|m \\ d=k \bmod N}} (-1)^{\frac{m}{d}-1} \right) e^{-\frac{2\pi\lambda_D}{N}m} . \tag{4.9}$$

This form of (4.7) exhibits the coefficients the instanton sum explicitly.

5. Condensates

The physical quantities which are computed from the eigenvalue distribution are the condensates $u_k = \mathrm{Tr}\Phi^k$. These essentially correspond to the integrals of motion for the integrable system, which are expressed as the symmetric polynomials of the eigenvalues of the Lax matrix.

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In this section we compare some of the condensates computed from the eigenvalues to earlier results for these. First we examine the semiclassical limit of the condensate $u_2 = \mathrm{Tr}\Phi^2$ in the Higgs vacuum.

The precise connection between u_2 and the symmetric polynomial constructed from the squares of the eigenvalues is encoded in the Lax equations. In order the Lax equations to give hamilton's equations, the square of the function $x(u)$ in the Lax matrix is related to the potential $V(u)$ by a constant, as described in (2.6). In the semiclassical limit

$$\begin{aligned} v_{ab} &= V(u_{ab}) = \frac{1}{4\sinh^2 \frac{1}{2}u_{ab}} + \frac{1}{12}\,, \\ x_{ab} &= X(u_{ab}) = \frac{1}{2}\coth\frac{1}{2}u_{ab} \end{aligned} \tag{5.1}$$

so in this limit

$$C = -\frac{1}{6}\,. \tag{5.2}$$

We also have

$$I_2 = \frac{1}{2}\mathrm{Tr}L^2 = \sum_{a>b} x_{ab}^2 = \sum_{a>b}(v_{ab} - C) \tag{5.3}$$

from which

$$u_2 = \sum_{a>b} v_{ab} = I_2 - \frac{N(N-1)}{12} = \frac{1}{2}\sum_{a=1}^{N}\lambda_a^2 - \frac{N(N-1)}{12}\,. \tag{5.4}$$

Substituting the semiclassical result for the eigenvalues (2.19)

$$u_2 = -\frac{N(N^2-1)}{24} \tag{5.5}$$

in perfect agreement with the semiclassical result of [14].

Next we compute the condensate u_2 in the confining vacuum. From the eigenvalues in the confining limit (3.6) we find

$$u_2 = I_2 = \frac{1}{2}\sum_{a=1}^{N}\lambda_a^2 = 2e^{-2\frac{\pi}{\lambda}}\sum_{a=1}^{N}\sin^2\frac{2a\pi}{N} = Ne^{-\frac{2\pi}{\lambda}} = Ne^{2\pi i\tau/N}\,. \tag{5.6}$$

This is in agreement with the result for u_2 computed in [14] in terms of Eisenstein series

$$U_2 = \frac{N^2}{24}\left(E_2(\tau) - \frac{1}{N}E_2\left(\frac{\tau}{N}\right)\right) \tag{5.7}$$

up to a vacuum independent shift function. The relevant part of U_2 is the second piece

$$U_2' = -\frac{N}{24}E_2\left(\frac{\tau}{N}\right) \approx Ne^{2\pi i\tau/N} \tag{5.8}$$

in agreement with u_2 computed from the eigenvalues.

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A. Distribution of vacua from the Seiberg-Witten curve

The Seiberg-Witten curve of the $\mathcal{N}=2^*$ SU(N) theory can be obtained from IIA/M-theory brane construction [10] where one considers a brane system suitably constructed from a single NS5 and N D4-branes. The Coulomb branch is obtained as a branched N-fold cover of the torus with complex structure τ

$$\tau = \frac{4i\pi}{g^2} + \frac{\theta}{2\pi} = \frac{iN}{\lambda} + \frac{\theta}{2\pi}\,, \tag{A.1}$$

where τ is the complex gauge coupling, and $\lambda = g^2 N/4\pi$ is the 't Hooft coupling.

The massive $\mathcal{N}=1^*$ vacua are the singular points on the Coulomb branch where the Seiberg-Witten curve maximally degenerates. At these points, the curve becomes an unbranched cover, a torus itself, with complex parameter

$$\bar{\tau} = \frac{p\tau + l}{q}\,, \qquad pq = N\,, \qquad l = 0\ldots q-1\,. \tag{A.2}$$

A generic vacuum is thus characterized by the 3 integers (p,q,l). The Higgs vacuum corresponds to $(N,1,0)$ while the confining vacuum is $(1,N,l)$. For the confining vacua, it is enough to consider the case $l=0$, because l nonzero is connected to this by a modular transformation.

The general SW curve can be described in terms of coordinates of the D4-branes as

$$\prod_{a=1}^{N} (v - v_a(z)) = 0 \tag{A.3}$$

here z is on the covered torus with complex structure τ, which has periods $\tau = \omega_2/\omega_1$.

The $\mathcal{N}=1^*$ vacua correspond to the singular points where the curve maximally degenerates to a torus with complex structure $\bar{\tau}$. The coordinates at these vacua, taking the case $l=0$, are given as [15]

$$\begin{aligned} v_{sr}(z) &= Nm\bar{\zeta}(z - z_1 + 2s\omega_1 + 2r\omega_2) - \\ &\quad - m\left(\sum_{t=0}^{p-1}\sum_{u=0}^{q-1} \bar{\zeta}(z + 2t\omega_1 + 2u\omega_2) + 2ps\bar{\zeta}(p\omega_1) + 2qr\bar{\zeta}(q\omega_2)\right), \end{aligned} \tag{A.4}$$

where $s = 0,1\ldots q-1$ and $r = 0,1\ldots p-1$. $\bar{\zeta}$ is the Weierstrass zeta function with complex structure $\bar{\tau}$ and periods $\bar{\omega}_1 = q\omega_1$, and $\bar{\omega}_2 = p\omega_2$, where we fix $\omega_1 = i\pi$. z_1 denotes the position of the NS5-brane, and m is the hypermultiplet mass.

In the following we determine the distribution of Higgs and confining vacua in the large-N limit.

In the Higgs vacuum (N, 1, 0) we denote the $r = 0,1\ldots N-1$ coordinates by $v_r^H(z)$

$$v_r^H(z) = Nm\bar{\zeta}(z - z_1 + 2ri\pi\tau) - m\left(\sum_{u=0}^{N-1} \bar{\zeta}(z + 2ui\pi\tau) + 2r\bar{\zeta}(Ni\pi\tau)\right). \tag{A.5}$$

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We are interested in the distribution of vacua around the their average, so we redefine the coordinates to be centered

$$V_r^h(z) = v_r^H(z) - \bar{v}^H(z) \tag{A.6}$$

so that

$$\sum_{r=0}^{N-1} V_r^H(z) = 0\,. \tag{A.7}$$

The new coordinates are then expressed as

$$V_r^h(z) = Nm\bar{\zeta}(z - z_1 + 2ri\pi\tau) - m\sum_{u=0}^{N-1}\bar{\zeta}(z - z_1 + 2ui\pi\tau) + m(N-1-2r)\bar{\zeta}(Ni\pi\tau)\,. \tag{A.8}$$

Taking the limit N large while keeping the 't Hooft coupling λ fixed, $\tau \to i\infty$. The $\bar{\zeta}$ function is periodic with periods

$$\begin{aligned}\bar{\omega}_1 &= q\omega_1 = i\pi\,,\\ \bar{\omega}_2 &= p\omega_2 = Ni\pi\tau\end{aligned} \tag{A.9}$$

and its complex structure is

$$\bar{\tau} = \frac{\bar{\omega}_2}{\bar{\omega}_1} = N\tau \longrightarrow i\infty\,. \tag{A.10}$$

In this limit the $\bar{\zeta}$ function reduces to

$$\bar{\zeta}(z) = -\frac{z}{12} + \frac{1}{2}\coth\frac{z}{2} \tag{A.11}$$

and

$$\begin{aligned}V_r^h(z) = {} & \frac{Nm}{2}\coth\left(\frac{z-z_1}{2} + ri\pi\tau\right) - \\ & -\frac{m}{2}\sum_{u=0}^{N-1}\coth\left(\frac{z-z_1}{2} + ui\pi\tau\right) + \frac{m}{2}(N-1-2r)\coth\frac{Ni\pi\tau}{2}\,.\end{aligned} \tag{A.12}$$

In the $\tau \to i\infty$ limit this becomes

$$\begin{aligned}V_r^h(z) &= mr - \frac{mN}{2} - \frac{m}{2}\coth\frac{z-z_1}{2}\,, \qquad r > 0\,,\\ V_0^h(z) &= mr + \frac{m(N-1)}{2}\coth\frac{z-z_1}{2}\,, \qquad r = 0\,.\end{aligned} \tag{A.13}$$

The fact that the $r = 0$ coordinate is different has no significance in the continuous $N \to \infty$ limit, since this is of measure zero.

To determine a distribution for the coordinates, we scale them between 0 and 1, that is divide by N. Defining the parameter

$$x = \frac{r}{N} \tag{A.14}$$

and

$$h(x) = \frac{V_r^h(z)}{N} \tag{A.15}$$

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we find

$$h(x) = mx - m/2\,. \tag{A.16}$$

Here the z-dependent piece, which is of order $1/N$, is dropped. Rescaling the coordinates by m we find the constant distribution

$$\rho_H(\tilde{h}) = \left.\frac{1}{\tilde{h}'(x)}\right|_{x=x(\tilde{h})} = 1\,. \tag{A.17}$$

Here we denoted the rescaled coordinates by $\tilde{h}$. (The mass m in the integrable system description is absorbed in the scaled variables in which the superpotential is written.)

The constant distribution computed from the curve agrees with the constant distribution found in terms of the eigenvalues of the Lax matrix in the semiclassical limit (2.19).

One can similarly analyze the supergravity limit of the confining (1, N, 0) vacua, with the centered coordinates

$$V_s^c(z) = Nm\bar{\zeta}(z - z_1 + 2si\pi) - m\sum_{t=0}^{N-1}\bar{\zeta}(z - z_1 + 2ti\pi) + m(N-1-2s)\bar{\zeta}(Ni\pi) \tag{A.18}$$

$\bar{\zeta}$ now has the periods

$$\begin{aligned} \bar{\omega}_1 &= Ni\pi\,, \\ \bar{\omega}_2 &= i\pi\tau \end{aligned} \tag{A.19}$$

and complex structure

$$\bar{\tau} = \frac{\bar{\omega}_2}{\bar{\omega}_1} = \frac{\tau}{N} \approx \frac{i}{\lambda}\,. \tag{A.20}$$

Here the θ term in the gauge coupling is neglected. In the large-N limit with fixed large 't Hooft coupling λ (the supergravity limit) $\bar{\tau} \to 0$. In order to get an expansion of the ζ function, a modular S-transformation has to be performed. The modular transformation acts on ζ as

$$\begin{aligned} \tau' &= \frac{a\tau + b}{c\tau + d}\,, \\ \zeta(z|\tau') &= (c\tau + d)\zeta(z(c\tau + d)|\tau)\,. \end{aligned} \tag{A.21}$$

Performing the S transformation on ζ when λ is large, so in the supergravity limit, the ζ function can be replaced by its semiclassical limit. Then the supergravity (large-N, large λ) limit can be similarly analyzed as for the case of the Higgs vacuum. We find a constant distribution in agreement with the distribution computed from the integrable system (4.8). Because of the modular properties of the ζ function, the coordinates of the Higgs and confining vacua as well as their large-N distributions are related by the modular transformation in $\tau = iN/\ \lambda$ as

$$-\frac{1}{\tau}V_s^h\left(-\frac{1}{\tau}, -\frac{1}{\tau}z\right) = V_s^c(\tau, z)\,. \tag{A.22}$$

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